JOURNAL OF COMPUTATIONAL PHYSICS 17, 1–9 (1975)

A Comparison of Three Two-Dimensional Electrostatic Plasma Simulation Models

H. R. LEWIS AND C. W. NIELSON

Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87544

Received August 6, 1973; revised August 8, 1974

Three two-dimensional plasma simulation models have been compared using an electrostatic two-stream instability as a test case. In one model the usual five-point Laplacian was used in Poisson's equation, and the field was smoothed by interpolation with a bilinear spline function. In another model the difference approximation to the Laplacian was the nine-point operator prescribed by Hamilton's variational principle, and the electric field was computed as the exact negative gradient of the potential. In the third model the nine-point Laplacian was used with the smoothed field. The test problem was an unstable double-streaming plasma that was initially cold in the streaming direction. The major result is that the evolution of electric field energy did not depend strongly on the choice of model. There was a much stronger dependence on the random numbers that were selected to represent the initial velocity distribution.

I. INTRODUCTION

We have compared three two-dimensional electrostatic plasma simulation models using as a test problem an unstable double-streaming electron plasma with an immobile neutralizing background charge distribution. The motivation for the comparison was to assess, for a problem in which collective effects dominate, the extent to which physical results depend on (a) the choice of numerical approximation of the two-dimensional Laplacian, and (b) smoothing the electric field as opposed to computing it as the exact negative gradient of the scalar potential at every point. In each model the potential was represented as a periodic bilinear spline function; that is, it was taken to be a continuous, periodic, piecewise linear function along each of two orthogonal coordinate axes. One model was that prescribed by Hamilton's variational principle when the potential is a periodic bilinear spline function [1, 2]. In that model the difference approximation to the Laplacian in Poisson's equation is a particular nine-point operator, and the electric field is computed as the exact negative gradient of the scalar potential. In another model the more usual five-point Laplacian was used, and the field

was smoothed by interpolation with a bilinear spline function. In the third model the nine-point Laplacian was used with the smoothed field. Our test problem was an unstable double-streaming plasma that was initially cold in the direction transverse to the streaming direction.

For each model we computed the electric energy as a function of time for two completely different sets of random numbers that were chosen to represent the initial velocity distribution; we also varied the number of cells and the cell length in the mesh on which Poisson's equation was solved. The result was that the evolution of the electric energy did not depend strongly on the choice of model for this problem; there was a much stronger dependence on the random numbers used for the initial loading.

We had hoped to demonstrate that one of the three simulation models is much better than the others in describing the evolution of electric energy in this example. However, the fact that differences in the evolution due to choice of model for a given initial loading were overshadowed by differences due to changes in the random initial loading is consistent with the results of two related studies [3, 4]. These two studies compared four one-dimensional simulation models with regard to collisional effects in a stable plasma [3], and with regard to the evolution of electric energy in an initially two-stream unstable plasma [4]. It was found, as in the present case, that certain quantities of physical interest are influenced more strongly by the choice of initial loading than by the choice of model.

These negative results in the search for better simulation methods do *not* mean that numerical simulation of plasmas is not useful or cannot be improved. However, they do mean that, to study *certain* interesting and important features of warm or hot plasma behavior, significant improvements must be made in our manner of representing distribution functions; the question of graininess in phase space is serious. It will not be adequate simply to represent phase-space distribution functions by a set of discrete particles of the sort currently used in simulations, nor will it suffice to improve our methods of solving Maxwell's equations numerically. More effective methods must be found for making *continuum* approximations of the plasma in terms of a finite set of parameters, and for describing complex aspects of plasma behavior phenomenologically. Denavit [5], and Knorr and Shoucri [6–8], have been concerned with the problem of effective continuum approximation recently, and Lewis [1] has described a general framework within which continuum approximations can be treated in terms of particle trajectories.

Our comparison of the three two-dimensional models is not exhaustive. There remain other interesting questions that should be studied, and it is to be hoped that someone will address them in the future. For example one can ask how the predictions of the models compare and contrast on other aspects of our test problem, as well as on other problems. Indeed, it would be worthwhile to compare

the models on a problem that is chosen to emphasize differences among them. We have restricted attention to an aspect of the evolution of an initially two-stream unstable plasma that seems to us to be of particular interest, and to represent a type of question of interest in plasma physics. Our results must not be construed as implying that all simulation models are equally good, only that in some intersting cases the choice is not crucial. Previous work by Langdon [9–11], Lindman [12], and Okuda [13] has revealed important differences in the dispersive and stability properties of various models. Most of their results are for one-dimensional, electrostatic models. Much further analysis and comparison is needed to complete our understanding of the properties and relative merits of simulation models, especially in two and three dimensions for fully electromagnetic models.

We now turn to the specific comparison that we have made. In Section II we describe the three simulation models and specify our test problem. The computational results are presented in Section III.

II. The Simulation Models and Test Problem

We consider N discrete particles in the periodic two-dimensional approximation. The particles move in the cartesian x–y plane only, the vector potential is identically zero, and the scalar potential depends only on the two spatial coordinates and on the time t. The scalar potential $\phi(x, y, t)$ in all three models is defined with respect to a square spatial grid consisting of M cells in each coordinate direction; the edge length of each cell is Δ, and the lines $x = 0$ and $y = 0$ are grid boundaries. The potential is taken to be continuous, piecewise linear in x and y, and periodic in x and y with period $L = M\Delta$. It is represented as

$$\phi(x, y, t) = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij}(t)\, g_i(x)\, g_j(y), \tag{1}$$

where the functions $g_n(x)$ are dimensionless local basis functions for one-dimensional, periodic, linear spline functions. They are defined as follows.

$$g_0(x) = \begin{cases} (1/\Delta)[\Delta - x], & \text{if } 0 \leqslant x \leqslant \Delta, \\ (1/\Delta)[x - (M-1)\,\Delta], & \text{if } (M-1)\,\Delta \leqslant x \leqslant M\Delta, \\ 0, & \text{otherwise,} \end{cases} \tag{2a}$$

$$g_n(x) = \begin{cases} (1/\Delta)[x - (n-1)\,\Delta], & \text{if } (n-1)\,\Delta \leqslant x \leqslant n\Delta, \\ (1/\Delta)[(n+1)\,\Delta - x], & \text{if } n\Delta \leqslant x \leqslant (n+1)\,\Delta, \\ 0, & \text{otherwise.} \end{cases} \tag{2b}$$

These functions are so-called "tent functions" of width 2Δ, and they are so defined that[1]

$$\phi(i\Delta, j\Delta, t) = \alpha_{ij}(t).$$

The difference approximations to Poisson's equation are of the form

$$-\sum_{n,m} T_{ij;nm}\alpha_{nm} = 4\pi P_{ij}\,, \tag{3}$$

where the matrix T is a representation of ∇^2, and P represents the charge density. For each model, the charge density is determined according to the usual "area-weighting" scheme. The explicit definition of P is

$$P_{ij} = \sum_{k=1}^{N} Q g_i(\gamma_k^{(x)})\, g_j(\gamma_k^{(y)}), \tag{4}$$

where $\gamma_k^{(x)}(t)$ and $\gamma_k^{(y)}(t)$ are, respectively, the x and y coordinates of particle k at time t, and Q is the charge of a particle.

The three models that we have compared are characterized as follows.

Model I. Use of the usual five-point ∇^2 operator with a smoothed $\mathbf{E}$. This model is similar to one used previously by Morse and Nielson [15][2].

Model II. Use of a particular nine-point ∇^2 operator with $\mathbf{E}$ computed directly from $\mathbf{E} = -\nabla\phi$. This is the prescription of Hamilton's variational principle.

Model III. Use of the nine-point ∇^2 operator from Model II with the smoothed $\mathbf{E}$ from Model I.

In Model I, the matrix T is the usual five-point operator for ∇^2; it is schematically represented by

$$\frac{1}{\Delta^2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{(five-point } \nabla^2\text{)}. \tag{5}$$

In Models II and III, the matrix T which represents ∇^2 is the nine-point operator

[1] $g_0(x)$ could also be taken equal to a constant. In pp. 413–418 of [2], an example is given in which the functions $g_n(x)$ are defined with $g_0(x) = 1$.

[2] Model II differs from that of Morse and Nielson in that the interpolation of E_x and E_y in Model II is from values at locations midway between grid lines instead of from values at locations on grid lines.

that is prescribed by the variational principle. A schematic representation of the nine-point operator is

$$\frac{1}{3\Delta^2}\begin{pmatrix}1 & 1 & 1\\ 1 & -8 & 1\\ 1 & 1 & 1\end{pmatrix} \qquad \text{(nine-point } \nabla^2\text{)}. \tag{6}$$

The corresponding matrix T is defined in terms of one-dimensional averaging and second derivative operators as follows.

$$T_{ij;nm} = u^{(2)}_{in}u^{(1)}_{jm} + u^{(1)}_{in}u^{(2)}_{jm}, \tag{7}$$

where

$$u^{(1)}_{in} = \int dx\, g_i(x)\, g_n(x), \tag{8a}$$

and

$$u^{(2)}_{in} = -\int dx\, g_i'(x)\, g_n'(x). \tag{8b}$$

Evaluation of these integrals yields:

$$u^{(1)}_{in} = \begin{cases}(\Delta/6)[4\delta_{in} + (\delta_{i-1,n} + \delta_{i+1,n})], & \text{if } i \neq 0 \text{ and } i \neq M-1,\\ (\Delta/6)[4\delta_{0,n} + (\delta_{M-1,n} + \delta_{1,n})], & \text{if } i = 0,\\ (\Delta/6)[4\delta_{M-1,n} + (\delta_{M-2,n} + \delta_{0,n})], & \text{if } i = M-1,\end{cases} \tag{9a}$$

$$u^{(2)}_{in} = \begin{cases}(1/\Delta)[-2\delta_{in} + (\delta_{i-1,n} + \delta_{i+1,n})], & \text{if } i \neq 0 \text{ and } i \neq M-1,\\ (1/\Delta)[-2\delta_{0,n} + (\delta_{M-1,n} + \delta_{1,n})], & \text{if } i = 0,\\ (1/\Delta)[-2\delta_{M-1,n} + (\delta_{M-2,n} + \delta_{0,n})], & \text{if } i = M-1.\end{cases} \tag{9b}$$

The operator $u^{(1)}$ is a (1, 4, 1) averaging operator, and $u^{(2)}$ is the central difference approximation to the second derivative.

The electric field components E_x and E_y are evaluated in two ways. In Model II they are simply the negative derivatives of ϕ, as prescribed by the variational principle:

$$E_x(x, y, t) = -\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}(t)\, g_i'(x)\, g_j(y), \tag{10a}$$

$$E_y(x, y, t) = -\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}(t)\, g_i(x)\, g_j'(y). \tag{10b}$$

In this case, $E_x(x, y, t)$ is a step function in x and piecewise linear in y, whereas $E_y(x, y, t)$ is a step function in y and piecewise linear in x. In Models I and III,

the field is *smoothed* in the following manner. $E_x(x, y, t)$ is linearly interpolated in x from the values $E_x[(n + 1/2)\,\Delta, y, t]$ determined from Eq. (10a); and $E_y(x, y, t)$ is linearly interpolated in y from the values $E_y[x, (n + 1/2)\,\Delta, t]$ determined from Eq. (10b). This smoothing amounts to bilinear interpolation of the components of **E**.

The electric field energy,

$$\begin{aligned} U &= \frac{1}{8\pi} \int dx\, dy\, \mathbf{E}^2(x, y, t) \\ &= \frac{1}{8\pi} \int dx\, dy\, \phi(x, y, t)\, \nabla^2\phi \end{aligned} \tag{11a}$$

is computed for Models I–III as

$$U = -\frac{1}{8\pi} \sum_{i,j} \sum_{n,m} \alpha_{ij} T_{ij;nm} \alpha_{nm} \,. \tag{11b}$$

The Test Problem

For the test runs whose results are presented in the next section, the initial loading consisted of two counterstreaming, unstable electron beams, each beam containing $119{,}025 = (345)^2$ particles. The distributions from which the initial velocities in the two beams were selected randomly are

$$\exp\left[-1/2\left(\frac{v_x - v_D}{v_{0x}}\right)^2 - 1/2\left(\frac{v_y}{v_{0y}}\right)^2\right]$$

and

$$\exp\left[-1/2\left(\frac{v_x + v_D}{v_{0x}}\right)^2 - 1/2\left(\frac{v_y}{v_{0y}}\right)^2\right].$$

The random numbers for the two beams were completely different. For all runs the values of the parameters were $v_D = \lambda_D\omega_p$, $v_{0x} = .01\lambda_D\omega_p$, and $v_{0y} = \lambda_D\omega_p$, where ω_p is the plasma frequency $(4\pi nQ^2/m)^{1/2}$, n being the total number density and m the mass per particle, and where λ_D is the Debye length corresponding to the thermal spread in v_y for either beam. The periodicity length L was either $32\lambda_D$ or $64\lambda_D$. Initially the particles in each beam were placed at the points of a uniform square grid consisting of 345×345 points. This grid was symmetrically situated within the grid on which the potential was defined. The separation in x or y between adjacent particles of either beam was $L/345$. M, the number of cells in x or y in the potential grid, was either 32 or 64. Two completely different random number sets were used to initialize the velocities for each combination of L, M,

and model. All runs wer made using the time-centered leapfrog scheme with a time step $\Delta t = .25\omega_p^{-1}$.

The initial velocities of the particles were all uncorrelated. Our initial loading does not represent a quiet start as defined by Byers [14], in which a given, single velocity distribution would be reproduced identically at each point of a spatial grid initially. Such a "quiet start" is very expensive in two and three spatial dimensions for hot plasmas. For example, if we had maintained the same number of particles per cell in the potential grid initially, but had used a quiet start with only ten initial velocities to represent a Maxwellian distribution, then we would have required *ten times* as many simulation particles.

There was no evidence of numerical instability with our initial loading. Total energy was well conserved for all runs: $(W_{\max} - W_{\min})/(W_{\max} + W_{\min})$ was less than 0.5%, where $W_{\max}$ and $W_{\min}$ are the maximum and minimum total energies, respectively. For Model II, the variational model, the fluctuation of total energy scaled as $(\Delta t)^2$, as expected.

III. Computational Results

Our example is an electron two-stream instability for which the plasma is initially rather cold in the streaming direction. The wavelength of most rapid linear growth is about $8\lambda_D$. Field energy histories are displayed in Fig. 1 and Fig. 2 for a variety of combinations of simulation parameters. The characteristic

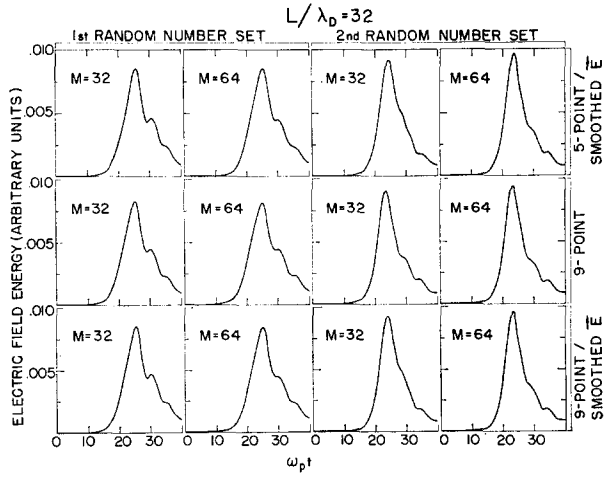

FIG. 1. Electrostatic field energy histories for cases with $L/\lambda_D = 32$.

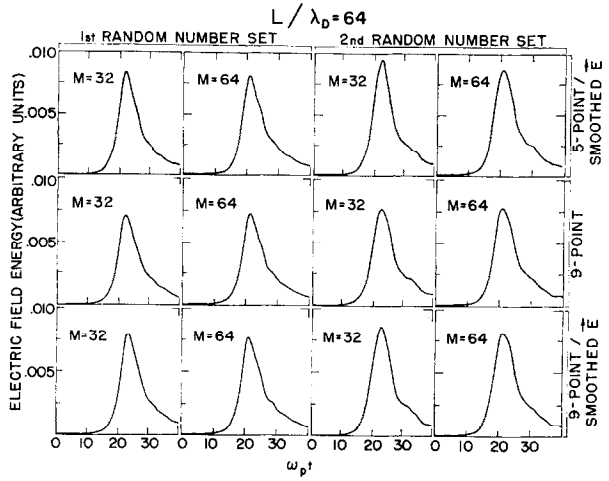

FIG. 2. Electrostatic field energy histories for cases with $L/\lambda_D = 64$.

rapid growth to a peak followed by a decay is exhibited in all cases. At the time of peak field energy, the plasma is dominated by a single mode, the fastest growing mode as determined by linear theory, and the evolution to a nearly flat-topped stable velocity distribution proceeds rapidly. These oscillations in the decay, most noticeable in Fig. 1 ($L/\lambda_D = 32$), correspond to the coalescence of trapped particle regions into regions of longer wavelength. This is the dominant process in one-dimensional simulations of the two-stream instability, but is of much less importance in two or three dimensions because trapping is not as effective [15]. In Figs. 1 and 2 the electric field energy is plotted vs time, instead of the logarithm of electric field energy vs time, in order to emphasize the differences among the various runs during the nonlinear phase after the initial peak in field energy.

The major conclusion that we draw from the field energy histories in Figs. 1 and 2, as well as from other diagnostics used in these simulations, such as phase space plots, is that some quantities of physical interest do not depend strongly on the choice of Model I, II, or III. We have observed here a much stronger dependence on the random number set used for the velocity initialization. It is conceivable that more dramatic model-dependent differences would be evident with some other test problem and other diagnostics.

ACKNOWLEDGMENTS

We thank L. Rudsinski for his programming assistance with these simulations. We also thank R. L. Morse for some helpful discussions.

REFERENCES

1. H. RALPH LEWIS, "Methods in Computational Physics," Vol. 9, p. 307, Academic Press, New York, 1970.
2. H. RALPH LEWIS, *J. Computational Phys.* **10** (1972), 400.
3. H. RALPH LEWIS, A. SYKES, AND J. A. WESSON, *J. Computational Phys.* **10** (1972), 85.
4. D. I. BROWN, S. J. GITOMER, AND H. R. LEWIS, *J. Computational Phys.* **14** (1974), 193.
5. J. DENAVIT, *J. Computational Phys.* **9** (1972), 75.
6. GEORG KNORR, *J. Computational Phys.* **13** (1973), 165.
7. GEORG KNORR AND MAGDI SHOUCRI, *J. Computational Phys.* **14** (1974), 1.
8. MAGDI SHOUCRI AND GEORG KNORR, *J. Computational Phys.* **14** (1974), 84.
9. A. BRUCE LANGDON, *J. Computational Phys.* **6** (1970), 247.
10. A. BRUCE LANGDON, *J. Computational Phys.* **12** (1973), 247.
11. A. BRUCE LANGDON, "Proceedings of the Fourth Conference on Numerical Simulation of Plasmas," Naval Research Laboratory, Washington, DC, November 2–3, 1970 (available from Superintendent of Documents, US Government Printing Office, Washington, DC 20402).
12. E. L. LINDMAN, *J. Computational Phys.* **5** (1970), 13.
13. H. OKUDA, *J. Computational Phys.* **10** (1972), 475.
14. J. A. BYERS, "Proceedings of the Fourth Conference on Numerical Simulation of Plasmas," Naval Research Laboratory, Washington, DC, November 2–3, 1970 (available from Superintendent of Documents, US Government Printing Office, Washington, DC 20402).
15. R. L. MORSE AND C. W. NIELSON, *Phys. Rev. Letters* **23** (1969), 1087.